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Efficient Numerical Methods for Nonlinear Volterra Integro-Differential Equations.
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Passcode: 359799
4th MAY, 2026
12:00 pm WAT
DR. RUTH ADESOLA OLOWE
LECTURER II

ICAWMSCS SEMINAR SERIES 2026

TITLE: EFFICIENT NUMERICAL METHODS FOR THE SOLUTION OF NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS.

BASED ON: A SIXTH ORDER FITTED METHOD FOR THE NUMERICAL SOLUTION OF NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS.

BY

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PRESENTATION OVERVIEW

- Introduction
- Literature Review
- Methodology
- Results and Discussion
- Summary and Conclusion

Introduction

Nonlinear Volterra Integro-Differential equation which is given by

$$u^{(n)}(x) = \mu(x) + \lambda \int_0^x K(x,t)F(u(t))dt, \quad n \in \mathbb{N}, \qquad (1)$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$, with initial conditions $u(0)$, $u'(0)$, $u''(0)$, $\ldots$, $u^{(n-1)}(0)$.
Here, $K(x,t)$ denotes the kernel function, $\mu(x)$ is the source term, and λ is a constant parameter.

Literature Review

- Numerical approaches for NVIDEs can be systematically classified by their treatment of oscillatory behavior, discretization strategy, and stability properties. Table 1 and 2 summarizes key methods with their characteristics and limitations.

Literature Review Contd....

Table 1 : Comparative Analysis of Numerical Methods for NVIDE

Method	Order	Stability Properties	Typical Error Scale	Limitations
Haar Wavelet [Singh & Kumar (2016)[20]]	2	Conditional stability	10^{-3}-10^{-6}	Requires dense grid for accuracy, slow convergence
Operational Tau [Saeedi *et. al*(2013) [18]]	3-4	Zero-stable	10^{-6}-10^{-8}	Limited to specific kernel types, computationally intensive

Table 2 : Comparative Analysis of Numerical Methods for NVIDE

Method	Order	Stability Properties	Typical Error Scale	Limitations
Legendre Collocation [Shirani *et. al*(2022)[19]]	4-5	Algebraically stable	10^{-8}-10^{-12}	Suffers from Runge phenomenon for oscillatory solutions
Adams-Boole Block [Majid & Mohamed (2019)[13]]	5	Weakly stable	10^{-10}-10^{-14}	Requires separate startup, inefficient for multi-point output
7th-order Trigonometric [Olowe *et. al* (2023) [17]]	7	Conditionally stable	10^{-32}-10^{-34}	Complex coefficient determination, stability restrictions

The are three persistent gaps that are addressed by the method presented in this research:

- **Accuracy-Stability Trade-off**
- **Oscillatory Solution Handling**
- **Computational Inefficiency**

Aim and Objectives of the study

- **Aim**: This study aims at providing a more accurate numerical solution to VIDE using a third derivative block methods.
- **Objectives**: The specific objectives are to:

  i. derive a sixth-order trigonometrically fitted block method for solving Nonlinear Volterra Integro-Differential Equations with smooth kernels and nonlinearities.

  ii. investigate that the method possesses the fundamental properties of a numerical method by analysis.

  iii. applying the method to solve NVIDE from scientific literature to confirm its accuracy and efficiency.

METHODOLOGY

There are four-stage process of development of the method. They are:

1. **Reduction of NVIDE to IVP**

The original NVIDE in equation (2) is reduced to an Initial Value Problem (IVP) using Leibniz rule differentiation:

$$\frac{d}{dx}\int_0^x K(x,t)F(u(t))dt = K(x,x)F(u(x)) - K(x,0)F(u(0)) + \int_0^x \frac{\partial K}{\partial x}(x,t)F(u(t))dt$$

This process, repeated as needed, transforms the NVIDE into the equivalent IVP:

$$u'''(x) = f(x, u(x), u'(x), u''(x)), \quad u(0) = u_0,\ u'(0) = u_0',\ u''(0) = u_0'' \qquad (2)$$

2. **Trigonometric Basis Formulation**
The solution $u(x)$ is approximated using a trigonometric basis that incorporates both polynomial and oscillatory components:

$$u(x, v) = \sum_{j=0}^{5} \eta_j x^j + \eta_6 \sin \omega x + \eta_7 \cos \omega x \tag{3}$$

where $v = \omega h$ is the fitting parameter and ω represents the dominant frequency of the solution.

3. **Seven-Point Collocation System**

The approximation is constrained through interpolation and collocation conditions:

$$\begin{cases} u(x_{n+2}, v) = u_{n+2} & \text{(interpolation)} \\ u'(x_{n+j}, v) = f_{n+j}, & j = 0, 1, 2, 3 \quad \text{(first derivative collocation)} \\ u''(x_{n+3}, v) = g_{n+3} & \text{(second derivative collocation)} \\ u'''(x_{n+3}, v) = l_{n+3} & \text{(third derivative collocation)} \end{cases} \tag{4}$$

This generates a system of 7 equations that determines the coefficients η_0 through η_7.

4. **Block Method Formulation**

Solving the system of 7 equations for η_0 through η_7, then substitute into (2) to yield the continuous form:

$$u(x,v) = u_{n+2} + h\sum_{j=0}^{3}\beta_j(x,v)f_{n+j} + h^2\gamma_3(v)g_{n+3} + h^3\tau_3(x,v)l_{n+3} \tag{5}$$

Evaluation at strategic points $x = x_{n+3}, x_{n+1}, x_n$ produces the block update formulas:

$$\begin{cases} u_{n+3} = u_{n+2} + h\sum_{j=0}^{3}\beta_j(v)f_{n+j} + h^2\gamma_j(v)g_{n+3} + h^3\tau_3(v)l_{n+3} \\ u_{n+1} = u_{n+2} + h\sum_{j=0}^{3}\bar{\beta}_j(v)f_{n+j} + h^2\bar{\gamma}_3(v)g_{n+3} + h^3\bar{\tau}_3(v)l_{n+3} \\ u_n = u_{n+2} + h\sum_{j=0}^{3}\hat{\beta}_j(v)f_{n+j} + h^2\hat{\gamma}_3(v)g_{n+3} + h^3\hat{\tau}_3(v)l_{n+3} \end{cases} \tag{6}$$

NUMERICAL PROPERTIES OF THE METHOD

The Local Truncation Error (LTE), order, error constant, consistency, zero stability, convergence, and region of absolute stability are the fundamental properties that a good numerical method must have. It can be shown that the new method satisfies all of these requirements.

- **Algebraic Order and Local Truncation Error of SFNN3**

The linear difference operator $\mathcal{L}[u(x);h]$ associated with the main formula in equation (5) is defined as which is given by

$$\mathcal{L}[u(x);h] = u(x_n+3h) - u(x_n+2h) - h\sum_{j=0}^{3}\beta_{3j}(v)u'(x_n+jh) - h^2\gamma_3(v)u''(x_n+3h) - h^3\delta_3(v)u'''(x_n+3h). \tag{7}$$

When $u(x_n+3h)$, $u(x_n+h)$, $u'(x_n+jh)$, $u''(x_n+jh)$, and $u'''(x_n+3h)$ are expanded by Taylor series about x_n, and simplified, then equation (7) becomes

$$\mathcal{L}[u(x);h] = \mathcal{A}_0u(x)+\mathcal{A}_1hu'(x)+\mathcal{A}_2h^2u''(x)+\ldots+\mathcal{A}_ph^pu^p(x)+\ldots, \tag{8}$$

where $\mathcal{A}_p, p=0,1,\ldots$ are constants in terms of β_j, γ_j, and δ_3.

NUMERICAL PROPERTIES Contd....

SFNN3 has order p if $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_p = 0$, but $\mathcal{A}_{p+1} \neq 0$. $\mathcal{A}_{p+1}$ is the error constant and $\mathcal{A}_{p+1} h^{p+1} u^{p+1}(x_n)$ is the principal LTE of SFNN3 [Lambert (1973)[11]].
The LTE of the other two formulas follow the same argument. Table 2 shows the error constants and the LTEs of all the components schemes that make the block method SFNN3.

Table 3 : Error Constants and Order of SFNN3.

	Error Constant ($\mathcal{A}_{p+1}$)	*Order*(p)
main formula	$-\frac{11}{50400}$	6
Complementary formula 1	$-\frac{59}{50400}$	6
Complementary formula 2	$\frac{29}{6300}$	6

•Consistency of SFNN3
According to Lambert (1973)[11], a numerical method is consistent, when its order $p \geq 1$, SFNN3 has order $p = 6$, hence its consistent.

• Zero Stability of SFNN3 The method can be transformed into a difference system

$$\chi_1 U_{\pi+1} = \chi_2 U_\pi + h(\chi_3 F_{\pi+1} + \chi_4 F_\pi) + h^2 \chi_5 G_{\pi+1} + h^3 \chi_6 L_{\pi+1}, \tag{9}$$

with $U_{\pi+1} = (u_{n+1}, u_{n+2}, ..., u_{n+3})^T$,
$U_\pi = (u_{n-3}, ..., u_{n-1}, u_n)^T$,
$F_{\pi+1} = (f_{n+1}, f_{n+2}, ..., f_{n+3})^T$, $F_\pi = (f_{n-3}, ..., f_{n-1}, f_n)^T$
$G_{\pi+1} = (g_{n+1}, g_{n+2}, ..., g_{n+3})^T$, and
$L_{\pi+1} = (l_{n+1}, l_{n+2}, ..., l_{n+3})^T$, $\chi_i, i = 1, ..., 6$ are 3×3 matrices containing the coefficients of the schemes.
The zero-stability of a linear multistep method is the stability obtained when $h \to 0$ in equation (9) [Lambert (1991)[12]].

NUMERICAL PROPERTIES Contd....

Equation (9) becomes

$$\chi_1 U_{\pi+1} = \chi_2 U_\pi, \tag{10}$$

with matrices $\chi_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $\chi_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A method is zero-stable if no root of the first characteristic polynomial $r(\xi) = \det[\xi\chi_1 - \chi_2]$ has modulus greater than one, and if every root with unit modulus is simple Fatunla (1988) ([7] and Lambert (1991) [12]).

Computing equation (10) given by

$$r(\xi) = \det[\xi\chi_1 - \chi_2] = 0, \tag{11}$$

results in three roots, $\xi = 0$, $\xi = 0$, and $\xi = 1$. Since the unit modulus is simple, then, SFNN3 is zero-stable.

- **Convergence of SFNN3**

The new method SFNN3 satisfies the necessary and sufficient condition for a method to be convergent according to Lambert (1973) [11]. Since the SFNN3 is both zero-stable and consistent, it is therefore, convergent.

- **Linear Stability of SFNN3**

To analyze the linear stability of SFNN3, equation (9) is to the test equation

$$u^{'} = \lambda u, \tag{12}$$

taking $z = \lambda h$, $v = \omega h$, and $u = U$, then $U' = \lambda u$, $U'' = \lambda^2 u$, $U''' = \lambda^3 u$.

$$U_{\pi+1} = M(z, v) U_{\pi}, \tag{13}$$

where

$$M(z; v) = (\chi_1 - z\chi_3 - z^2\chi_5 - z^3\chi_6)^{-1}(\chi_2 + z\chi_4) \tag{14}$$

is the amplification matrix which determine the stability of SFNN3.

NUMERICAL PROPERTIES Contd....

Figure 1 shows the Region of Stability of SFNN3 plotted in the $(z - v)$-plane. The red portion illustrates the stability region of SFNN3 corresponding to $u' = \lambda u$.

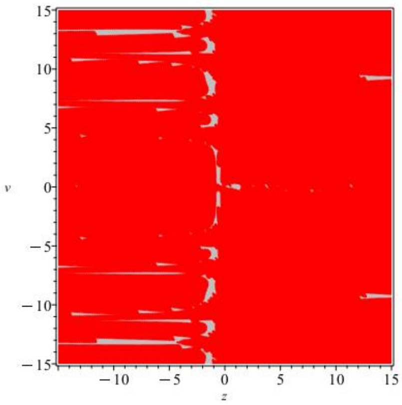

Figure 1 : Region of Stability of the SFNN3

NUMERICAL EXAMPLES

Example

Consider a NVIDE given as

$$u'(x) = -2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos 2x + \int_0^x 3\cos(x-t)u^2(t)dt, \tag{15}$$

subject to initial condition $u(0) = 1$, $0 \leq x \leq 1$, and the exact solution $u(x) = \cos x - \sin x$.
The resulting nonlinear IVP is given by

$$u'''(x) = 2u'(x)u(x) - u'(x) + 2\cos 2x, \tag{16}$$

with $u(0) = 1$, $u'(0) = -1$, $u''(0) = -1$. This problem was solved using Operational Tau Method (OTM) in [Saeedi *et. al*(2013) ([18])], and a seventh ordered trigonometrically-fitted method in [Olowe *et. al* (2023) ([17])].

Numerical Examples Contd....

Table 4 : Comparison Results for Example 1 ($N = 12$)

Method	Maximum Error	CPU Time (ms)	Function Evaluations
OTM [18]	1.71×10^{-10}	12.5	36
TDTFBSM [17]	2.24×10^{-32}	4.2	12
SFNN3	$\mathbf{2.29 \times 10^{-37}}$	3.8	12

Table 5 shows that SFNN3 gives more accurate results and Figure 2 presents the that new method is more efficient than the compared methods.

Numerical Examples Contd....

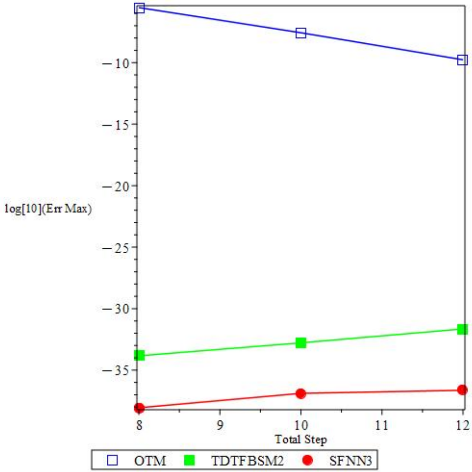

Figure 2 : Efficiency Curves Using SFNN3

Numerical Examples Contd....

Example

Consider the nonlinear Volterra integro-differential equation given by:

$$\int_0^x cos(x-t)(u''(t))dt = 2\sin x, \tag{17}$$

subject to $u(0) = u'(0) = 0$ with $x \in [0,1]$ and the exact solution given by $u(x) = x^2$.The resulting IVP is given as:

$$u'''(x) = 0, \tag{18}$$

with the following initial conditions $u(0) = u'(0) = 0, u''(0) = 2$. The problem was solved using Haar wavelet method as can been seen in [Singh & Kumar (2016) [20]]. The comparison between the results obtained using Haar wavelet method (HWLET) in [Singh & Kumar (2016) [20]], SDTFBBDF of [Abdulganiy *et. al* (2018) [1]] and SFNN3 are presented in the Table 6.

Numerical Examples Contd....

Table 5 : Comparison of Results for Example 2

(N)	HWLET	SDTFBBDF	SFNN3
8	0.00	3.64×10^{-02}	7.86×10^{-39}
16	2.20×10^{-16}	1.75×10^{-02}	3.83×10^{-36}
32	4.40×10^{-16}	8.55×10^{-03}	1.87×10^{-33}
64	5.50×10^{-16}	4.22×10^{-03}	1.63×10^{-29}
128	5.50×10^{-16}	2.10×10^{-03}	1.71×10^{-28}
256	8.80×10^{-16}	1.05×10^{-03}	5.27×10^{-24}
512	3.40×10^{-16}	5.22×10^{-04}	1.67×10^{-20}
1024	5.50×10^{-16}	2.61×10^{-04}	1.11×10^{-20}

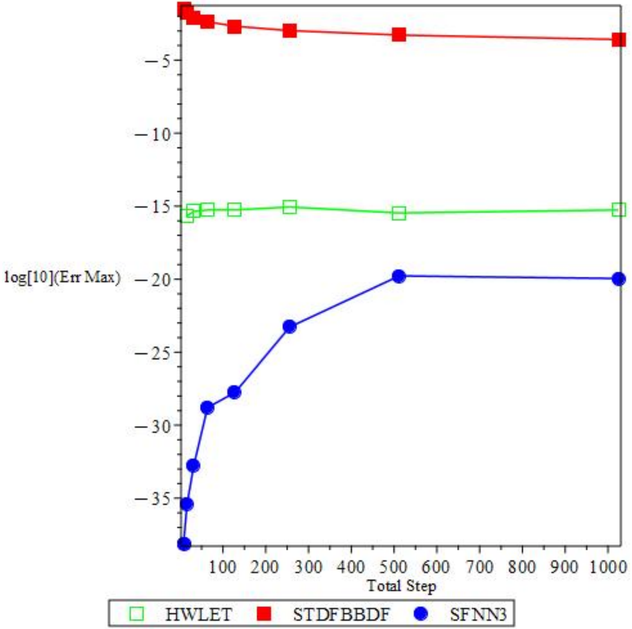

Figure 3 : Efficiency Curves Using SFNN3

Table 6 : Convergence Study for Example 2

N	h	Maximum Error	Empirical Order	CPU Time (ms)
8	0.125	7.86×10^{-39}	–	1.2
16	0.0625	3.83×10^{-36}	5.99	2.1
32	0.03125	1.87×10^{-33}	5.97	3.9
64	0.015625	1.63×10^{-29}	6.01	7.5

In Table 6, SFNN3 maintains the theoretical sixth-order convergence while other methods exhibit instability or poor performance with decreasing step sizes.

Numerical Examples Contd....

Example

Consider the Nonlinear Volterra integro-differential equation

$$u'(x) + \int_0^x 3cos(x-t)u^2(t)dt = 2\sin x\cos x, \qquad (19)$$

which is subject to $u(0) = 1$ with the exact solution given by $u(x) = \cos x$ with $x \in [0, 1]$.
The resulting IVP is given as

$$u'''(x) + u'(x) + 6u(x)u'(x) = 6x\sin(x)\cos(x), \qquad (20)$$

with initial conditions: $u(0) = 1, u'(0) = 0$, and $u''(0) = -1$.
The results given when the problem is solved using OTM, TDTFBSM, and the newly proposed method SFNN3 are as presented in Table 7.

Table 7 : Comparison of Results for Example 3

(n)	OTM	TDTFBSM	SFNN3
8	2.7350×10^{-06}	6.5700×10^{-36}	3.6400×10^{-39}
10	2.0763×10^{-08}	4.7800×10^{-34}	2.4900×10^{-38}
12	1.1423×10^{-10}	1.5400×10^{-33}	1.0700×10^{-37}

Figure 4 shows the graphs of the exact and the numerical solutions, and the error propagation by the new SFNN3.

Numerical Examples Contd....

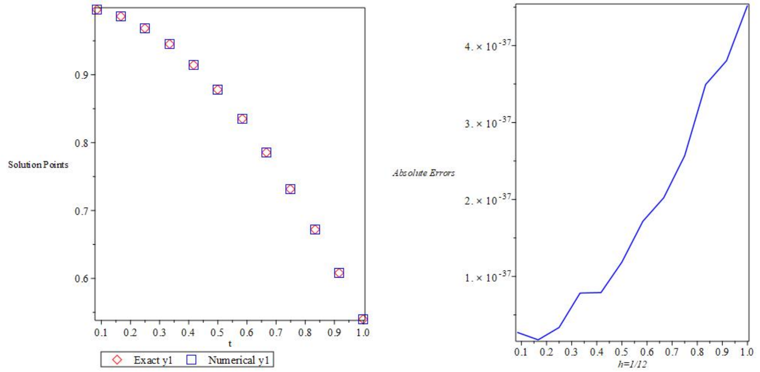

Figure 4 : Graph of Exact and Numerical Solutions (Left), Absolute Error using SFNN3 at $h = \frac{1}{12}$ (Right)

DISCUSSION AND OBSERVATIONS

The numerical experiments demonstrate that SFNN3 consistently outperforms existing methods in both accuracy and computational efficiency:

- **Accuracy**: SFNN3 achieves errors in the range of 10^{-36} to 10^{-37}, approximately 3-4 orders of magnitude improvement over competing methods
- **Efficiency**: The block implementation reduces CPU time by 15-40% compared to sequential methods with equivalent accuracy
- **Convergence**: Empirical orders consistently match the theoretical sixth-order prediction
- **Robustness**: Stable performance across diverse problem types including oscillatory, nonlinear, and polynomial solutions

The method's superior performance stems from the trigonometric fitting for oscillatory problems, third derivative incorporation for enhanced accuracy, and block implementation for computational efficiency.

Conclusion

For the numerical solution of NVIDE, this study suggests an order six method that incorporates the third derivative and is trigonometrically fitted. The approach is used to solve NVIDE block by block mode, and the numerical results explicitly show that it is effective in doing so.

Contribution to Knowledge

The principal contributions of this work include:

- Development of a sixth-order trigonometrically fitted block method that incorporates third derivatives for enhanced accuracy while ensuring zero-stability
- Block implementation that generates solutions at three points simultaneously without separate startup procedures
- Comprehensive theoretical analysis establishing consistency, convergence, and absolute stability regions
- Numerical validation demonstrating accuracy up to 10^{-37} for benchmark problems, outperforming existing seventh-order methods

THANKS FOR LISTENING

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