Numerical Solution of Linear and Nonlinear Higher-order Initial Value Problems Using Optimized Hybrid Block Schemes

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PRESENTATION OUTLINE

- Introduction
- Literature Review
- Methodology
- Analysis of Developed Method
- Numerical Results and Discussion
- Conclusion
- Selected References

1.1. Background of the study

- **Differential Equations**

Historically, the concept of Differential Equations originated from the work of Isaac Newton on dynamics in the 17th century. Differential Equations (DE) are used to model real life problems that relates the rate of change of various quantities to their current values to predict future behaviour. Examples of DE are population growth and decay, Newton second law of motion, spread of epidemic, etc.

⋆ **The order of an ODE** is the highest derivative present in the ODE.

⋆ If there is no product of the dependent variable with itself or any of its derivatives, then the equation is said to be **linear**, otherwise it is **nonlinear**.

INTRODUCTION (Contd.)

- **Higher-order ODE**

⋆ The general form of an mth-order ODE is given by

$$g(t, y, y', y'', ..., y^{(m)}) = 0, \quad t \in [t_0, t_N] \tag{1}$$

also written as

$$y^{(m)}(t) = g(t, y, y', y'', ..., y^{(m-1)}), \quad t \in [t_0, t_N]$$

where m represents higher-order derivatives and y and its derivatives are functions of t and t_0 and t_N are the start and end points of the integration interval.

⋆ **A general solution** of the ODE(1) is a relation between y, t, and m arbitrary constants which satisfies the equation but which contains no derivatives. The general solution of the ODE can be written in an explicit form

$$y = w(t, c_1, c_2, ..., c_m) \tag{2}$$

The m arbitrary constants can be determined by prescribing m **conditions** of the form

$$y^{(u)}(t_0) = y_0^{(u)}, \quad u = 0, 1, 2, 3, ..., m-1. \tag{3}$$

at a point $t = t_0$ which are called **Initial Conditions**. The point t_0 is called Initial point. The DE (1) with the Initial Conditions (3) is called an **mth-order Initial Value Problem**.

The general **mth-order Initial Value Problem** is of the form

$$y^{(m)}(t) = g(t, y, y', y'', ..., y^{(m-1)}), \quad t \in [t_0, t_N]$$

subject to Initial conditions

$$y^{(u)}(t_0) = y_0^{(u)}, \quad u = 0, 1, 2, 3, ..., m-1. \qquad (4)$$

Equation (4) can be solved analytically but do not always have exact solutions hence the need for numerical methods to proffer an approximate solution for $y(t)$. Also the Lipschitz Existence and Uniqueness Theorem is assumed satisfied (Jain et al. (1984)).

Examples of Higher-order IVP (4) are

$\star$ Second-order: Vibrations problem, Newton second law of motion

$\star$ Third-order: nonlinear Blasius problem in Fluid Dynamics, also these can be used to model motion of rocket, quantum mechanics, chemical engineering.

$\star$ Fourth-order: Lumped system parameters in Vibrations, these are also used to model beam theory, electric circuits, neural network.

$\star$ Fifth-order: these are seen in viscoelastic flow

Though some of the above problems are Boundary Value Problems or Initial-Boundary Value Problems when subject to boundary conditions.

Existing Methods for Higher-order IVPs

• Higher-order IVPs can be solved either by first reducing the given IVP to systems of first-order IVP and then apply numerical methods for First-order IVPs or solved directly without reduction (see Lambert, 1973, 1991).

• Numerical methods for higher-order IVPs are either single step methods or Multistep methods (Boutayeb and Chetouani, 2007). These are:

⋆ Runge- Kutta method: classical and modified (see Butcher, 2003)

⋆ Predictor-Corrector schemes like Adams-Bashforth predictor and Adams-Moulton corrector schemes (Audu et al. 2020,)

⋆ Finite Difference schemes (Twizell (1986, 1987), Boutayeb and Chetouani (2007))

LITERATURE REVIEW (Contd)

⋆ Modified Collocation Methods like Hybrid Block methods (Akinnukawe and Odekunle (2023)), Five-step Block method (Adesanya et al. (2012)), Implicit Block Method (Adeyeye et al.(2019)), Algorithmic Collocation Method (Awoyemi (2005)), Multistep method (Jator(2008)), Zero-stable Block method (Kuboye et al.(2020)) to mention few.

- Approximate Analytical Methods (Semi-Analytical Methods) like

⋆ Differential Transformation Method (Ogwumu et al. (2020), Alaa Khatib(2016))

⋆ Adomian Decomposition Method (Adomian (1988), Wazwaz (2002))

, ⋆ Homotopy Analysis Method (Al-Hayani and Fahad (2019), Liao (1992))

⋆ Variational Iteration Method (Baghdadi and Ahammad (2024), Wazwaz (2007)) and so on

Methodology

• A Single-step method is a numerical method in which the solution at any point is obtained using the solution only at the previous point. Single step methods can be either explicit of the form

$$y_{j+1} = y_j + hf(t_j, y_j, h)$$

or implicit of the form

$$y_{j+1} = y_j + hf(t_j, y_j, t_{j+1}, y_{j+1}, h)$$

• A Multi-step method uses the values of $y(t)$ and $y'(t)$ at $(k+1)$ successive mesh/grid points to determine the value of $y(t)$ at the point t_{j+1}.

• The Hybrid method is a numerical method that involves the introduction of off-step points in the derivation process of the method to circumvent the **Second Dalquist Barrier Theorem** for solving the given IVP. This method is usually implemented in the predictor-corrector mode (Dalquist (1963),Butcher (2003)).

Methodology (Contd)

• The Block method is a numerical method that when applied to an IVP produces approximate solution at multi points simultaneously. The Block method was first used by Milne (1953) for use only as starting values for predictor-corrector algorithm.
• In this research work, the properties of both the Hybrid and the Block methods are brought together. The Hybrid Block method preserve the Runge-kutta traditional advantage of being self starting. They are more accurate than the R-K method because the Hybrid method is implemented in a Block form.
These single-step and Multistep methods can be developed through collocation and interpolation techniques (see Onumanyi et al., 1994) using the power series as its basis function.

Derivation of An optimized One-step Hybrid Block Method

The solution of equation (4) where $m = 4$ is assumed on an interval $[t_n, t_{n+1}]$ which is locally approximated by a polynomial of the form

$$y(t) = \sum_{r=0}^{(p_1+p_2-1)} \phi_r t^r, \tag{5}$$

with corresponding derivatives as

$$y'(t) = \sum_{r=1}^{p_1+p_2-1} r\phi_r t^{r-1}, \tag{6a}$$

$$y''(t) = \sum_{r=2}^{p_1+p_2-1} r(r-1)\phi_r t^{r-2}, \tag{6b}$$

$$y'''(t) = \sum_{r=3}^{p_1+p_2-1} r(r-1)(r-2)\phi_r t^{r-3}, \tag{6c}$$

where p_1 and p_2 are the interpolation and collocation points respectively. The two hybrid points (w_1 and w_2) are considered in such a way that $0 < w_1 < w_2 < 1$ holds. Interpolating equation (5) and collocating equation (6) at given grid points gives

$$y_{n+j} = y(t_{n+j}), \;\; j = 0,$$

$$y'_{n+j} = y'(t_{n+j}), \;\; j = 0,$$

$$y''_{n+j} = y''(t_{n+j}), \;\; j = 0,$$

$$y'''_{n+j} = y'''(t_{n+j}), \;\; j = 0,$$

$$y^{(iv)}_{n+j} = g(t_{n+j}), \;\; j = 0, w_1, w_2, 1, \quad (7)$$

Derivation Contd

where y_{n+j} and g_{n+j} are approximations for $y(t_{n+j})$ and $y^{(iv)}(t_{n+j})$ respectively. The system of eight equations in equation (7) is written in compact form as $yA = G$ where

$$y = \begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 \\ 0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 \\ 0 & 0 & 0 & 0 & 24 & 120t_n & 360t_n^2 & 840t_n^3 \\ 0 & 0 & 0 & 0 & 24 & 120t_{n+w_1} & 360t_{n+w_1}^2 & 840t_{n+w_1}^3 \\ 0 & 0 & 0 & 0 & 24 & 120t_{n+w_2} & 360t_{n+w_2}^2 & 840t_{n+w_2}^3 \end{bmatrix}$$

Derivation Contd

$$A = [\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7]^T \qquad (8b)$$

and

$$G = \left[y_n, y_n', y_n'', y_n''', g_n, g_{n+w_1}, g_{n+w_2}, g_{n+1}\right]^T \qquad (8c)$$

.

Solving equation (8) simultaneously gives the corresponding coefficients of $\phi_r, r = 0(1)7$. Substituting the resulting coefficients $\phi_r, r = 0(1)7$ into equation (5) and its derivatives yields a continuous implicit scheme of the form,

$$\alpha_z y_{n+z} = \alpha_0 y_n + h\beta_{10} y_n' + h^2 \beta_{20} y_n'' + h^3 \beta_{30} y_n''' + h^4 \sum_{j=0}^{1} \rho_j g_{n+j}$$

$$+ h^4 \sum_{j=1}^{2} \rho_{w_j} g_{n+w_j}, z = w_1, w_2, 1. \qquad (9)$$

To obtain the approximate values of w_1 and w_2 hybrid points, optimize the local truncation errors of one of the schemes in equation (9) and ensure that the hybrid points satisfy the interval $0 < w_1 < w_2 < 1$, specifically we choose y'_{n+1} (Rufai and Ramos (2020)). To get the local truncation error, expand the Taylor series about the point t_n of the scheme to obtain

$$L[y'(t_{n+1}), h] = h^{q+4}y^{(q+4)}(t_n)C_{q+4} + O(h^9) = \frac{h^8 y^{(8)}(t_n)V_1}{20160} + O(h^9) \tag{10}$$

where C_{q+4} is the Error constant of the particular scheme y'_{n+1},

$$V_1 = -21w_1w_2 + 7w_1 + 7w_2 - 3 = 0, \tag{11a}$$

$$0 < w_1 < w_2 < 1. \tag{11b}$$

Imposing that the principal term (V_1) in the local truncation error (10) is zero. Solve $(11a)$ and its constraint $(11b)$ to obtain w_1 and w_2 and this yields the possible solution as $w_1 = \frac{1}{4}$ and $w_2 = \frac{5}{7}$. The discrete schemes and its derivatives derived by evaluating (9) at grid and non-grid points $\left(\frac{1}{4}, \frac{5}{7}, 1\right)$ are given in equations $(12) - (15)$. These schemes are used to form a block of hybrid method and its derivative methods as:

$$\begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{5}{7}} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} y_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{5}{7} \\ 1 \end{bmatrix} \begin{bmatrix} hy'_n \end{bmatrix} + \begin{bmatrix} \frac{1}{32} \\ \frac{25}{98} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} h^2 y''_n \end{bmatrix}$$

Derivation Contd

$$+\begin{bmatrix} \frac{1}{384} \\ \frac{125}{2058} \\ \frac{1}{6} \end{bmatrix} \left[\; h^3 y_n''' \; \right] + \begin{bmatrix} 0 & 0 & \frac{217}{1843200} \\ 0 & 0 & \frac{129125}{29647548} \\ 0 & 0 & \frac{11}{900} \end{bmatrix} \begin{bmatrix} h^4 g_{n-\frac{5}{7}} \\ h^4 g_{n-\frac{1}{4}} \\ h^4 g_n \end{bmatrix}$$

$$+\begin{bmatrix} \frac{1}{18432} & -\frac{49}{3686400} & \frac{1}{245760} \\ \frac{50000}{7411887} & -\frac{125}{302526} & \frac{3125}{19765032} \\ \frac{16}{585} & \frac{49}{23400} & 0 \end{bmatrix} \begin{bmatrix} h^4 g_{n+\frac{1}{4}} \\ h^4 g_{n+\frac{5}{7}} \\ h^4 g_{n+1} \end{bmatrix}. \qquad (12)$$

The first, second and third derivative block methods are in equations (13) - (15) respectively.

$$\begin{bmatrix} hy'_{n+\frac{1}{4}} \\ hy'_{n+\frac{5}{7}} \\ hy'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} hy'_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{5}{7} \\ 1 \end{bmatrix} \begin{bmatrix} h^2 y''_n \end{bmatrix} + \begin{bmatrix} \frac{1}{32} \\ \frac{25}{98} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} h^3 y'''_n \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \frac{1063}{614400} \\ 0 & 0 & \frac{3875}{201684} \\ 0 & 0 & \frac{11}{300} \end{bmatrix} \begin{bmatrix} h^4 g_{n-\frac{5}{7}} \\ h^4 g_{n-\frac{1}{4}} \\ h^4 g_n \end{bmatrix}$$

$$+\begin{bmatrix} \frac{311}{299520} & -\frac{3773}{15974400} & \frac{53}{737280} \\ \frac{80000}{1966419} & \frac{125}{214032} & \frac{625}{2420208} \\ \frac{64}{585} & \frac{343}{15600} & -\frac{1}{720} \end{bmatrix} \begin{bmatrix} h^4 g_{n+\frac{1}{4}} \\ h^4 g_{n+\frac{5}{7}} \\ h^4 g_{n+1} \end{bmatrix}. \qquad (13)$$

$$\begin{bmatrix} h^2 y''_{n+\frac{1}{4}} \\ h^2 y''_{n+\frac{5}{7}} \\ h^2 y''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left[h^2 y''_n \right] + \begin{bmatrix} \frac{1}{4} \\ \frac{5}{7} \\ 1 \end{bmatrix} \left[h^3 y'''_n \right]$$

$$+\begin{bmatrix} 0 & 0 & \frac{229}{12800} \\ 0 & 0 & \frac{1525}{28812} \\ 0 & 0 & \frac{7}{100} \end{bmatrix} \begin{bmatrix} h^4 g_{n-\frac{5}{7}} \\ h^4 g_{n-\frac{1}{4}} \\ h^4 g_n \end{bmatrix}$$

Derivation Contd

$$+\begin{bmatrix} \frac{581}{37440} & -\frac{1029}{332800} & \frac{43}{46080} \\ \frac{50000}{280917} & \frac{425}{15288} & \frac{625}{172872} \\ \frac{176}{585} & \frac{343}{2600} & -\frac{1}{360} \end{bmatrix} \begin{bmatrix} h^4 g_{n+\frac{1}{4}} \\ h^4 g_{n+\frac{5}{7}} \\ h^4 g_{n+1} \end{bmatrix}. \qquad (14)$$

$$\begin{bmatrix} h^3 y'''_{n+\frac{1}{4}} \\ h^3 y'''_{n+\frac{5}{7}} \\ h^3 y'''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h^3 y'''_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{391}{3840} \\ 0 & 0 & \frac{55}{1029} \\ 0 & 0 & \frac{1}{15} \end{bmatrix} \begin{bmatrix} h^4 g_{n-\frac{5}{7}} \\ h^4 g_{n-\frac{1}{4}} \\ h^4 g_n \end{bmatrix}$$

Derivation Contd

$$+\begin{bmatrix} \frac{103}{624} & -\frac{2401}{99840} & \frac{11}{1536} \\ \frac{2000}{4459} & \frac{265}{1092} & -\frac{125}{4116} \\ \frac{16}{39} & \frac{343}{780} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} h^4 g_{n+\frac{1}{4}} \\ h^4 g_{n+\frac{5}{7}} \\ h^4 g_{n+1} \end{bmatrix}. \qquad (15)$$

Equations (12) – (15) form the Novel Fourth order One-step Block Algorithm with optimal Hybrid points (NFOBA) developed for the direct approximation of linear and non-linear fourth-order IVP (4).

- **The Error Constant and Order of the Method**

The Local truncation error associated with the developed method can be defined as the linear difference operator (see Akinnukawe and Odekunle (2023))

$$L[y(t_n);h] = \sum_{j=0}^{1} \alpha_j y(t_n+jh) - \sum_{i=1}^{3} h^{(i)} \beta_{i0} y^{(i)}(t_n) - h^4 \sum_{j=0}^{1} \rho_j y^{(iv)}(t_n+jh)$$

$$-h^4 \sum_{j=1}^{2} \rho_{w_j} y^{(iv)}(t_n+(w_j)h). \qquad (16)$$

Assuming that $y(t_n)$ is sufficiently differentiable, then using Taylor series expansion on $y^{(i)}(t_n+jh), i = 0(1)4$ about t_n, we have

Analysis of The Developed Method (NFOBA)

$$y(t_n + jh) = \sum_{q=0}^{\infty} \frac{(jh)^q}{q!} y^{(q)}(t_n),$$

$$y'(t_n + jh) = \sum_{q=0}^{\infty} \frac{(jh)^q}{q!} y^{(q+1)}(t_n),$$

$$y''(t_n + jh) = \sum_{q=0}^{\infty} \frac{(jh)^q}{q!} y^{(q+2)}(t_n),$$

$$y'''(t_n + jh) = \sum_{q=0}^{\infty} \frac{(jh)^q}{q!} y^{(q+3)}(t_n),$$

$$y^{(iv)}(t_n + jh) = \sum_{q=0}^{\infty} \frac{(jh)^q}{q!} y^{(q+4)}(t_n),$$

Substituting $y(t_n + jh)$, $y'(t_n + jh)$, $y''(t_n + jh)$, $y'''(t_n + jh)$ and $y^{(iv)}(t_n + jh)$ in equation (16) to obtain

Analysis of The Developed Method (NFOBA)

$$L[y(t_n); h] = C_0 y(t_n) + C_1 h y'(t_n) + C_2 h^2 y''(t_n)$$

$$+ C_3 h^3 y'''(t_n) + \dots + C_{q+4} h^{q+4} y^{(q+4)}(t_n) + \dots \qquad (17)$$

where $C_q, q = 0, 1, 2, \dots$ are constants given as:

$$C_0 = \sum_{j=0}^{1} \alpha_j + \sum_{j=1}^{2} \alpha_{w_j}$$

$$C_1 = \left[\sum_{j=0}^{1} j\alpha_j + \sum_{j=1}^{2} w_j \alpha_{w_j} \right] - \beta_{10}$$

$$\vdots$$

$$C_{q+4} = \frac{1}{(q+4)!}\left[\sum_{j=0}^{1} j^{q+4}\alpha_j + \sum_{j=1}^{2}(w_j)^{q+4}\alpha_{w_j}\right] - \frac{1}{(q+3)!}\left[\sum_{j=0}^{1} j^{q+3}\beta_{1j} \right.$$

$$-\frac{1}{(q+2)!}\left[\sum_{j=0}^{1} j^{q+2}\beta_{2j} + \sum_{j=1}^{2}(w_j)^{q+2}\beta_{2w_j}\right] - \frac{1}{(q+1)!}\left[\sum_{j=0}^{1} j^{q+1}\beta_{3j} + \sum_{j=1}^{2} \right.$$

$$-\frac{1}{q!}\left[\sum_{j=0}^{1} j^{q}\rho_j + \sum_{j=1}^{2}(w_j)^{q}\rho_{w_j}\right] \tag{18}$$

A numerical method for solving fourth-order IVP is said to be of order q if $C_0 = C_1 = C_2 = \ldots = C_{q+1} = C_{q+2} = C_{q+3} = 0$, and $C_{q+4} \neq 0$, then C_{q+4} is called the Error constant of the method. Using equation (18), the error constants and order of convergence of NFOBA are shown in the next table (Table 1).

Analysis of The Developed Method (NFOBA)

S/N	Scheme	Order(q)	Error Constant (C_{q+4})
1	$y_{n+\frac{1}{4}}$	4	$-\frac{1877}{55490641920}$
2	$y_{n+\frac{5}{7}}$	4	$-\frac{53125}{34865516448}$
3	y_{n+1}	4	$-\frac{1}{423360}$
4	$y'_{n+\frac{1}{4}}$	4	$-\frac{13}{22020096}$
5	$y'_{n+\frac{5}{7}}$	4	$-\frac{8125}{1660262688}$
6	y'_{n+1}	5	$\frac{1}{705600}$
7	$y''_{n+\frac{1}{4}}$	4	$-\frac{311}{41287680}$
8	$y''_{n+\frac{5}{7}}$	4	$\frac{625}{135531648}$
9	y''_{n+1}	4	$\frac{1}{40320}$
10	$y'''_{n+\frac{1}{4}}$	4	$-\frac{83}{1474560}$
11	$y'''_{n+\frac{5}{7}}$	4	$\frac{2125}{19361664}$
12	y'''_{n+1}	4	$\frac{1}{40320}$

- **Zero-stability**

The numerical method is said to be zero-stable if the roots of the first characteristic polynomial $\gamma(R)$ satisfy $|R_v| \leq 1, v = 1, \ldots, 12$ multiplicity not exceeding the order of the differential equation (Akinnukawe et al. (2024)). The first characteristic polynomial $\gamma(R) = 0$ of the derived method is calculated as

$$\gamma(R) = \det(RA_{(1)} - A_{(0)})$$

where $A_{(1)}$ is a 12 by 12 identity matrix and

$$A_{(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{32} & 0 & 0 & \frac{1}{384} \\ 0 & 0 & 1 & 0 & 0 & \frac{5}{7} & 0 & 0 & \frac{25}{98} & 0 & 0 & \frac{125}{2058} \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{5}{7} & 0 & 0 & \frac{25}{32} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{5}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

$\gamma(R) = R^{v-e}(R-1)^e$ where e is the order of the differential equation and v is the order of the matrices $A_{(1)}$ and $A_{(0)}$. The NFOBA can be shown to be zero-stable since the first characteristic polynomial $\gamma(R) = R^8(R-1)^4$ satisfies $|R_v| \leq 1, v = 1(1)12$.

- **Consistency**

The developed method is concluded to be consistent since according to Lambert (1991), the necessary and sufficient condition for a numerical scheme to be consistent is for it to have order of at least one ($q \geq 1$). The derived method is of order 4 since the least order of the block method is of order 4.

- **Convergence**

A numerical method converges if it is consistent and zero-stable. This implies that NFOBA converges since the method is of order $q = 4 > 1$ and it satisfies the conditions for zero-stability.

The following problems are considered in order to examine the accuracy and computational efficiency of the new block method (NFOBA). All computations are done using MATHEMATICA 13.0. The notations used in representing the existing methods and the derived method in the result Tables are:

h	step siz
EIFO	Error in (Familua and Omole, 2017
EIFBM	Error of order 7 in First Block Method (Kuboye et al., 2020
EIBBCM	Error in (Akinnukawe and Odekunle, 2023
EINFOBA	Error in Novel Fourth order One-step Block Algorithr

where,

$$Error^{(i)} = |y_{exact}^{(i)} - y_{appro}^{(i)}|$$

These Problems are chosen to aid comparison with other existing methods in literature (Kuboye et al. (2020)).

Problem 1

$$y^{(iv)}(t) - (y')^2 + yy''' + 4t^2 - e^t(1 - 4t + t^2) = 0, \quad h = \frac{0.1}{32};$$

$$y(0) = 1, \ \ y'(0) = 1, \ \ y''(0) = 3,$$
$$y'''(0) = 1.$$

The theoretical solution is

$$y(t) = t^2 + e^t.$$

Table 2 Comparison of Errors for Problem 1

t	EIFBM	EIFO	EINFOBA
0.103125	$1.8149E-10$	$9.0214E-10$	$2.2204E-16$
0.206250	$1.1543E-08$	$1.2168E-09$	$1.3322E-15$
0.309375	$1.2194E-07$	$1.2168E-09$	$1.1102E-15$
0.412500	$6.5296E-07$	$1.7137E-09$	$1.9984E-15$
0.515625	$2.3972E-06$	$1.4819E-08$	$3.3306E-15$
0.618750	$6.7092E-06$	$3.0583E-08$	$2.6645E-15$
0.721875	$1.6438E-05$	$4.9418E-08$	$8.8818E-16$
0.825000	$3.5549E-05$	$7.1286E-08$	$7.1054E-15$
0.928125	$6.9845E-05$	$1.0587E-07$	$1.8207E-14$
1.031250	$1.2716E-04$	$1.4455E-07$	$2.5313E-14$

Problem 2

Consider the Ship Dynamic Problem

$$y^{(iv)}(t) + 3y'' + y(2 + \rho cos(wt)) = 0, \quad t > 0,$$
$$y(0) = 1, y'(0) = y''(0) = y'''(0) = 0.$$

When $\rho = 0$, the theoretical solution is

$$y(t) = 2cos(t) - cos(\sqrt{2}t).$$

Table 3 Comparison of Errors for Problem 2

t	EIBBCM	EINFOBA ($h = 0.01$)
3	$9.29840E-08$	$1.20615E-12$
6	$3.08597E-07$	$7.22977E-13$
9	$4.19093E-07$	$1.15525E-11$
12	$9.32249E-08$	$2.27867E-11$
15	$4.97079E-07$	$1.75494E-11$

Problem 3

$$y^{(iv)}(t) - y''' - y'' - y' - 2y = 0, \quad t \in [0, 2],$$

$$y(0) = y'(0) = y''(0) = 0,$$

$$y'''(0) = 30,$$

Exact solution is

$$y(t) = 2exp(2t) - 5exp(-t) + 3cos(t) - 9sin(t).$$

Table 4 Numerical Results for Problem 3

t	Exact	NFOBA	EINFOBA
0.2	0.042171386260	0.042171386260	$1.0151E-14$
0.4	0.357899528037	0.35789952803	$1.8207E-13$
0.6	1.290400249177	1.290400249177	$1.0491E-12$
0.8	3.293335338149	3.293335338151	$3.8049E-12$
1.0	6.98638304634	6.98638304634	$1.0791E-11$
1.2	13.2391031914	13.2391031914	$2.6249E-11$
1.4	23.29716258129	23.29716258131	$5.7621E-11$
1.6	38.9718168099	38.97181681001	$1.1754E-10$
1.8	62.92373948426	62.9237394843	$2.2713E-10$
2.0	99.0875062990	99.08750629921	$4.2132E-10$

Numerical Results and Discussions

The developed Block scheme (NFOBA) is efficient and effective in the numerical Integration of the fourth-order linear and nonlinear Initial Value Problems of Ordinary Differential Equations as seen in Tables 2 - 4. NFOBA was compared with existing methods used to solve same problems and it shows superiority to these methods.

Conclusion

An effective one-step block hybrid method for solving linear and non-linear fourth-order initial value problems of ordinary differential equations is derived and applied directly. The introduced hybrid points w_1 and w_2 in the derived scheme are computed in such a way that the points lies in the interval $0 < w_1 < w_2 < 1$. The approximate values of the hybrid points were obtained by optimizing the local truncation error of one of the derived scheme. The application of NFOBA on three problems shows the efficiency of the new block method when compared to some existing methods. The analysis of the method were shown to be consistent, zero-stable and convergent. NFOBA has proven effective in the direct integration of fourth-order Initial value problems of ordinary differential equations. Furthermore, the derived method can also be applied to Boundary Value problems (BVP) by first converting the BVP to IVP using variable transformation method.

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Thank you for listening